

# CAFFARELLI-KOHN-NIRENBERG TYPE INEQUALITIES OF FRACTIONAL ORDER WITH APPLICATIONS

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ABSTRACT. Let  $0 < s < 1$  and  $p > 1$  be such that  $ps < N$ . Assume that  $\Omega$  is a bounded domain containing the origin. Staring from the ground state inequality by R. Frank and R. Seiringer in [16] to obtain:

- (1) The following improved Hardy inequality for  $p \geq 2$ :

For all  $q < p$ , there exists a positive constant  $C \equiv C(\Omega, q, N, s)$  such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy - \Lambda_{N,p,s} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p} dx \geq C \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}} dx dy$$

for all  $u \in C_0^\infty(\mathbb{R}^N)$ . Here  $\Lambda_{N,p,s}$  is the optimal constant in the Hardy inequality (1.2).

- (2) Define  $p_s^* = \frac{pN}{N-ps}$  and let  $\beta < \frac{N-ps}{2}$ , then

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps} |x|^\beta |y|^\beta} dy dx \geq S(N, p, s, \beta) \left( \int_{\mathbb{R}^N} \frac{|u(x)|^{p_s^*}}{|x|^{2\beta \frac{p_s^*}{p}}} dx \right)^{\frac{p}{p_s^*}},$$

for all  $u \in C_0^\infty(\Omega)$  where  $S \equiv S(N, p, s, \beta) > 0$ .

- (3) If  $\beta \equiv \frac{N-ps}{2}$ , as a consequence of the improved Hardy inequality, we obtain that for all  $q < p$ , there exists a positive constant  $C(\Omega)$  such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps} |x|^\beta |y|^\beta} dy dx \geq C(\Omega) \left( \int_{\Omega} \frac{|u(x)|^{p_{s,q}^*}}{|x|^{2\beta \frac{p_{s,q}^*}{p}}} dx \right)^{\frac{p}{p_{s,q}^*}},$$

for all  $u \in C_0^\infty(\Omega)$  where  $p_{s,q}^* = \frac{pN}{N-qs}$ .

Notice that the previous inequalities can be understood as the fractional extension of the Caffarelli-Kohn-Nirenberg inequalities in [9].

## 1. INTRODUCTION

In [9] the authors proved the following result

**Theorem 1.1** (Caffarelli-Kohn-Nirenberg). *Let  $p, q, r, \alpha, \beta, \sigma$  and  $a$  be real constants such that  $p, q \geq 1$ ,  $r > 0$ ,  $0 \leq a \leq 1$ , and*

$$\frac{1}{p} + \frac{\alpha}{N}, \frac{1}{q} + \frac{\beta}{N}, \frac{1}{r} + \frac{m}{N} > 0,$$

*where  $m = a\sigma + (1-a)\beta$ . Then there exists a positive constant  $C$  such that for all  $u \in C_0^\infty(\mathbb{R}^N)$  we have*

$$\left\| |x|^m u \right\|_{L^r(\mathbb{R}^N)} \leq C \left\| |x|^\alpha |\nabla u| \right\|_{L^p(\mathbb{R}^N)}^a \left\| |x|^\beta u \right\|_{L^q(\mathbb{R}^N)}^{1-a},$$

*if and only if the following relations hold:*

$$\frac{1}{r} + \frac{m}{N} = a \left( \frac{1}{p} + \frac{\alpha-1}{N} \right) + (1-a) \left( \frac{1}{q} + \frac{\beta}{N} \right),$$

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with

$$0 \leq \alpha - \sigma \text{ if } a > 0,$$

and

$$\alpha - \sigma \leq 1 \text{ if } a > 0 \text{ and } \frac{1}{r} + \frac{m}{N} = \frac{1}{p} + \frac{\alpha - 1}{N}.$$

This class of inequalities are related to the following local elliptic problem

$$(1.1) \quad -\operatorname{div}(|x|^{-p\gamma}|\nabla u|^{p-2}\nabla u) = 0.$$

As a consequence of Theorem 1.1, it follows that  $|x|^{-\gamma}$ , with  $\gamma < \frac{N-p}{p}$ , is an admissible weight in the sense that if  $u$  is a weak positive supersolution to (1.1), then it satisfies a weak Harnack inequality.

More precisely, there exists a positive constant  $\kappa > 1$  such that for all  $0 < q < \kappa(p-1)$ ,

$$\left( \int_{B_{2\rho}(x_0)} u^q(x) |x|^{-p\gamma} dx \right)^{\frac{1}{q}} \leq C \inf_{B_\rho(x_0)} u,$$

where  $B_{2\rho}(x_0) \subset \subset \Omega$ , and  $C > 0$  depends only on  $B$ .

We refer to [13], [19] and the references therein for a complete discussion and the proof of the Harnack inequality.

Notice that even the classical Harnack inequality holds for positive solution to (1.1).

One of the main tools to get the weak Harnack inequality is a weighted Sobolev inequality that can be obtained directly from Theorem 1.1.

An alternative argument to get the Sobolev inequality is to prove a weighted Hardy inequality as it was observed in [22].

The main goal of this paper is to follow this approach in order to get a nonlocal version of the Caffarelli-Kohn-Nirenberg inequalities.

In [16], the authors proved the following Hardy inequality stating that for  $p > 1$  with  $sp < N$  and for all  $\phi \in C_0^\infty(\mathbb{R}^N)$ ,

$$(1.2) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{N+ps}} dx dy \geq \Lambda_{N,p,s} \int_{\mathbb{R}^N} \frac{|\phi(x)|^p}{|x|^{ps}} dx$$

where the constant  $\Lambda_{N,p,s}$  is given by

$$(1.3) \quad \Lambda_{N,p,s} = 2 \int_0^\infty |1 - \sigma^{-\gamma}|^{p-2} (1 - \sigma^{-\gamma}) \sigma^{N-1} K(\sigma) d\sigma$$

and

$$K(\sigma) = \int_{|y'|=1} \frac{dH^{n-1}(y')}{|x' - \sigma y'|^{N+ps}}.$$

In the same paper, and setting

$$h_s(u) \equiv \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy - \Lambda_{N,p,s} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx,$$

they proved that for  $p \geq 2$ , there exists a positive constant  $C = C(p, N, s)$  such that for all  $u \in C_0^\infty(\mathbb{R}^N)$ , if  $v = |x|^{\frac{N-ps}{p}} u$ , then

$$(1.4) \quad h_s(u) \geq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^{\frac{N-ps}{2}}} \frac{dy}{|y|^{\frac{N-ps}{2}}}.$$

The above inequality turns to be equality for  $p = 2$  with  $C = 1$ .

As a consequence of (1.4), we easily get that  $\Lambda_{N,p,s}$  is never achieved.

For  $p = 2$ , the authors in [2] proved the next result:

**Theorem 1.2.** *Let  $N \geq 1$ ,  $0 < s < 1$  and  $N > 2s$ . Assume that  $\Omega \subset \mathbb{R}^N$  is a bounded domain, then for all  $1 < q < 2$ , there exists a positive constant  $C = C(\Omega, q, N, s)$  such that for all  $u \in C_0^\infty(\Omega)$ ,*

$$(1.5) \quad a_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - \Lambda_{N,2,s} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} dx \geq C(\Omega, q, N, s) \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+qs}} dx dy.$$

One of the main results of this work is to generalize the result of Theorem 1.2 to the case  $p > 2$ . More precisely we have the next Theorem:

**Theorem 1.3.** *Let  $p > 2$ ,  $0 < s < 1$  and  $N > ps$ . Assume that  $\Omega \subset \mathbb{R}^N$  is a bounded domain, then for all  $1 < q < p$ , there exists a positive constant  $C = C(\Omega, q, N, s)$  such that for all  $u \in C_0^\infty(\Omega)$ ,*

$$(1.6) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy - \Lambda_{N,p,s} \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx \geq C \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}} dx dy.$$

As a consequence we get the next "fractional" Caffarelli-Kohn-Nirenberg inequality in bounded domain.

**Theorem 1.4.** *Let  $p \geq 2$ ,  $0 < s < 1$  and  $N > ps$ . Assume that  $\Omega \subset \mathbb{R}^N$  is a bounded domain, then for all  $1 < q < p$ , there exists a positive constant  $C = C(\Omega, q, N, s)$  such that for all  $u \in C_0^\infty(\Omega)$ ,*

$$(1.7) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} dx dy \geq C \left( \int_{\Omega} \frac{|u(x)|^{p_{s,q}^*}}{|x|^{2\beta \frac{p_{s,q}^*}{p}}} dx \right)^{\frac{p}{p_{s,q}^*}}$$

where  $p_{s,q}^* = \frac{pN}{N-qs}$  and  $\beta = \frac{N-ps}{2}$ .

In the case where  $\Omega = \mathbb{R}^N$ , to get a *natural* generalization of the classical Caffarelli-Kohn-Nirenberg inequality obtained in [9], we have to consider a class of *admissible weights* in the sense of [19]. Precisely we obtain the following weighted Sobolev inequality.

**Theorem 1.5.** *Assume that  $1 < p < \frac{N}{s}$  and let  $0 < \beta < \frac{N-ps}{2}$ , then for all  $u \in C_0^\infty(\mathbb{R}^N)$ , we have*

$$(1.8) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} \geq S(\beta) \left( \int_{\mathbb{R}^N} \frac{|u(x)|^{p_{s,q}^*}}{|x|^{2\beta \frac{p_{s,q}^*}{p}}} dx \right)^{\frac{p}{p_{s,q}^*}},$$

where  $S(\beta) > 0$ .

It is clear that the condition imposed on  $\beta$  coincides in some sense with definition of *admissible* weight given in [19]. The proof of Theorem 1.5 is based on some weighted Hardy inequality given below.

As a direct application of the previous results, we will consider the problem

$$(1.9) \quad \begin{cases} L_{p,s} u & - \lambda \frac{|u|^{p-2} u}{|x|^{ps}} = |u|^{q-1} u, & u > 0 \text{ in } \Omega, \\ u & = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where

$$L_{s,p} u(x) := \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+ps}} dy,$$

$0 < \lambda \leq \Lambda_{N,p,s}$  and  $q > 0$ .

In the local case, the problem is reduced to

$$(1.10) \quad \begin{cases} -\Delta_p u & - \lambda \frac{|u|^{p-2} u}{|x|^p} = |u|^{q-1} u, & u > 0 \text{ in } \Omega, \\ u & = 0 & \text{in } \partial\Omega. \end{cases}$$

For  $p = 2$ , the authors in [7] proved that if  $q > q_+(2)$ , then problem (1.10) has no distributional supersolution, however, if  $q < q_+(2)$ , there exists a positive supersolution, with  $q_+(2) = 1 + \frac{2}{\theta_1}$ ,  $\theta_1 = \frac{N-2}{2} - \sqrt{\Lambda_{N,2} - \lambda}$  and  $\Lambda_{N,2} = \frac{(N-2)^2}{4}$ , the classical Hardy constant.

The case  $p \neq 2$  was considered in [1] where the same alternative holds with  $q_+(p) = p - 1 + \frac{p}{\theta_p}$  where  $\theta_p$  is the smallest solution to the equation

$$\Xi(s) = (p-1)s^p - (N-p)s^{p-1} + \lambda.$$

The fractional case with  $p = 2$  was studied in [14] and [5]. The authors proved the same alternative with  $q_+(2, s) = 1 + \frac{2s}{\theta}$  where  $\theta \equiv \theta(\lambda, s, N) > 0$ .

Our goal is to extend the results of [14] and [5] to the case  $p \neq 2$ .

The paper is organized as follows.

In Section 2 we prove the main results, namely Theorems 1.3, 1.4 and 1.5.

The starting point will be the proof of a general version of the Picone inequality. As a consequence, we get a weighted version of the Hardy inequality for a class of "admissible weights".

Hence, following closely the arguments used in [2], taking in consideration the "weighted" Hardy inequality, we get the proof of Theorem 1.3.

Once Theorem 1.3 proved, we complete the proof of Theorem 1.4 using suitable Sobolev inequality.

At the end, and by using a weighted Hardy inequality, we are able to get a "fractional Caffarelli-Kohn-Nirenberg" inequality for admissible weights in  $\mathbb{R}^N$  and then to proof Theorem 1.5.

In section 3, we analyze problem (1.10). We prove the existence of a critical exponent  $q_+(p, s)$  such that if  $q > q_+(p, s)$ , then problem (1.10) has no positive solution in a suitable sense. To show the optimality of the non-existence exponent, we will construct an appropriate supersolution in the whole space.

In the whole of the paper we will use the next elementary inequality, see for instance [16].

**Lemma 1.6.** *Assume that  $p > 1$ , then for all  $0 \leq t \leq 1$  and  $a \in \mathbb{C}$ , we have*

$$(1.11) \quad |a - t|^p \geq (1 - t)^{p-1}(|a|^p - t).$$

## 2. STATEMENT AND PROOF OF THE MAIN RESULTS

Let us begin with some functional settings that will be used below, we refer to [12] and [22] for more details.

For  $s \in (0, 1)$  and  $p \geq 1$ , we define the fractional Sobolev spaces  $W^{s,p}(\Omega)$ ,  $\Omega \subset \mathbb{R}^N$ , by

$$W^{s,p}(\Omega) \equiv \{u \in L^p(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy < \infty\}.$$

It is clear that  $W^{s,p}(\Omega)$  is a Banach space endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{\frac{1}{p}}.$$

In the same way, we define the space  $X_0^{s,p}(\Omega)$  as the completion of  $C_0^\infty(\Omega)$  with respect to the norm of  $W^{s,p}(\Omega)$ .

Notice that, if  $Q = \mathbb{R}^N \times \mathbb{R}^N \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ , then

$$\|\phi\|_{X_0^{s,p}(\Omega)} = \left( \int_Q \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{\frac{1}{p}} + \|\phi\|_{L^p(\Omega)}.$$

Using the fractional Sobolev inequality we obtain  $X_0^{s,p}(\Omega) \subset L^{p_s^*}(\Omega)$  with continuous inclusion, where  $p_s^* = \frac{pN}{N-sp}$  for  $ps < N$ .

In the case where  $\Omega$  is a bounded regular domain, the space  $X_0^{s,p}(\Omega)$  can be endowed with the equivalent norm

$$|||\phi|||_{X_0^{s,p}(\Omega)} = \left( \int_Q \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{\frac{1}{p}}.$$

To prove the *fractional* Caffarelli-Kohn-Nirenberg inequality, we need to define fractional Sobolev spaces with weight. More precisely, let  $0 < \beta < \frac{N-ps}{2}$  and  $\Omega \subset \mathbb{R}^N$  with  $0 \in \Omega$ , the weighted Sobolev space  $X^{s,p,\beta}(\Omega)$  is defined by

$$X^{s,p,\beta}(\Omega) := \left\{ \phi \in L^p(\Omega, \frac{dx}{|x|^{2\beta}}) : \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{N+ps}} \frac{dx dy}{|x|^{\beta}|y|^{\beta}} < +\infty \right\}.$$

Thus  $X^{s,p,\beta}(\Omega)$  is a Banach space endowed with the norm

$$\|\phi\|_{X^{s,p,\beta}(\Omega)} = \left( \int_{\Omega} \frac{|\phi(x)|^p dx}{|x|^{2\beta}} \right)^{\frac{1}{p}} + \left( \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{N+ps}} \frac{dx dy}{|x|^{\beta}|y|^{\beta}} \right)^{\frac{1}{p}}.$$

Now, we define the weighted Sobolev space  $X_0^{s,p,\beta}(\Omega)$  as the completion of  $C_0^\infty(\Omega)$  with respect to the previous norm.

As in [3], see also [12], we can prove the following extension result.

**Lemma 2.1.** *Assume that  $\Omega \subset \mathbb{R}^N$  is a regular domain, then for all  $w \in X^{s,p,\beta}(\Omega)$ , there exists  $\tilde{w} \in X^{s,p,\beta}(\mathbb{R}^N)$  such that  $\tilde{w}|_{\Omega} = w$  and*

$$||\tilde{w}||_{X^{s,p,\beta}(\mathbb{R}^N)} \leq C ||w||_{X^{s,p,\beta}(\Omega)}$$

where  $C \equiv C(N, s, p, \Omega) > 0$ .

**Remark 2.2.** As in the case  $\beta = 0$ , if  $\Omega$  is bounded regular domain, we can endow  $X_0^{s,p,\beta}(\Omega)$  with the equivalent norm

$$|||\phi|||_{X_0^{s,p,\beta}(\Omega)} = \left( \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{N+ps}} \frac{dx dy}{|x|^{\beta}|y|^{\beta}} \right)^{\frac{1}{p}}.$$

Now, for  $w \in X_0^{s,p,\beta}(\Omega)$ , we set

$$L_{s,p,\beta}(w)(x) = \text{P.V.} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^{p-2} (w(x) - w(y))}{|x - y|^{N+ps}} \frac{dy}{|x|^{\beta}|y|^{\beta}}.$$

It is clear that for all  $w, v \in X^{s,p,\beta}(\mathbb{R}^N)$ , we have

$$\langle L_{s,p,\beta}(w), v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^{p-2} (w(x) - w(y)) (v(x) - v(y))}{|x - y|^{N+ps}} \frac{dx dy}{|x|^{\beta}|y|^{\beta}}.$$

In the case where  $\beta = 0$ , we denote  $L_{s,p,\beta}$  by  $L_{s,p}$ .

Let begin by proving the next version of the Picone inequality.

**Lemma 2.3.** (*Picone inequality*) *Let  $w \in X_0^{s,p,\beta}(\Omega)$  be such that  $w > 0$  in  $\Omega$ . Assume that  $L_{s,p,\beta}(w) = \nu$  with  $\nu \in L_{loc}^1(\mathbb{R}^N)$  and  $\nu \geq 0$ , then for all  $u \in C_0^\infty(\Omega)$ , we have*

$$\frac{1}{2} \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx dy}{|x|^{\beta}|y|^{\beta}} \geq \langle L_{s,p,\beta} w, \frac{|u|^p}{w^{p-1}} \rangle.$$

*Proof.* The case  $p = 2$  and  $\beta = 0$  was obtained in [21] and [6] for  $p \neq 2$ . For the reader convenience we include some details for the case  $\beta \neq 0$ .

We set  $v(x) = \frac{|u(x)|^p}{|w(x)|^{p-1}}$  and  $k(x, y) = \frac{1}{|x - y|^{N+ps}|x|^\beta|y|^\beta}$ , then

$$\begin{aligned} \langle L_{s,p,\beta}(w(x)), v(x) \rangle &= \int_{\Omega} v(x) \int_{\mathbb{R}^N} |w(x) - w(y)|^{p-2} (w(x) - w(y)) k(x, y) dy dx \\ &= \int_{\Omega} \frac{|u(x)|^p}{|w(x)|^{p-1}} \int_{\mathbb{R}^N} |w(x) - w(y)|^{p-2} (w(x) - w(y)) k(x, y) dy dx. \end{aligned}$$

Since  $k$  is symmetric, we obtain that

$$\begin{aligned} \langle L_{s,p,\beta}(w(x)), v(x) \rangle &= \\ \frac{1}{2} \int \int_Q \left( \frac{|u(x)|^p}{|w(x)|^{p-1}} - \frac{|u(y)|^p}{|w(y)|^{p-1}} \right) |w(x) - w(y)|^{p-2} (w(x) - w(y)) k(x, y) dy dx. \end{aligned}$$

Let  $v_1 = \frac{u}{w}$ , then

$$\begin{aligned} \langle L_{s,p,\beta}(w(x)), v(x) \rangle &= \\ \frac{1}{2} \int \int_Q (|v_1(x)|^p w(x) - |v_1(y)|^p w(y)) |w(x) - w(y)|^{p-2} (w(x) - w(y)) k(x, y) dy dx. \end{aligned}$$

Define

$$\Phi(x, y) = |u(x) - u(y)|^p - (|v_1(x)|^p w(x) - |v_1(y)|^p w(y)) |w(x) - w(y)|^{p-2} (w(x) - w(y)),$$

then

$$\begin{aligned} \langle L_{s,p,\beta}(w(x)), v(x) \rangle &+ \frac{1}{2} \int_Q \Phi(x, y) k(x, y) dy dx \\ &= \frac{1}{2} \int \int_Q |u(x) - u(y)|^p k(x, y) dy dx. \end{aligned}$$

We claim that  $\Phi \geq 0$ . It is clear that, by a symmetry argument, we can assume that  $w(x) \geq w(y)$ . Let  $t = w(y)/w(x)$ ,  $a = u(x)/u(y)$ , then using inequality (1.11), the claim follows at once. Hence we conclude.  $\square$

As a consequence, for  $\beta = 0$ , we have the next comparison principle that extends, to the fractional framework, the classical one obtained by Brezis-Kamin in [8].

**Lemma 2.4.** *Let  $\Omega$  be a bounded domain and let  $f$  be a nonnegative continuous function such that  $f(\sigma) > 0$  if  $\sigma > 0$  and  $\frac{f(\sigma)}{\sigma^{p-1}}$  is decreasing. Let  $u, v \in W_0^{s,p}(\Omega)$  be such that  $u, v > 0$  in  $\Omega$  and*

$$\begin{cases} L_{s,p}u & \geq f(u) \text{ in } \Omega, \\ L_{s,p}v & \leq f(v) \text{ in } \Omega, \end{cases}$$

Then,  $u \geq v$  in  $\Omega$ .

*Proof.* Using an approximation argument, taking in consideration that  $u, v > 0$ , we can prove that

$$(2.12) \quad \frac{L_{s,p}u}{u^{p-1}} - \frac{L_{s,p}v}{v^{p-1}} \geq \left( \frac{f(u)}{u^{p-1}} - \frac{f(v)}{v^{p-1}} \right).$$

We set  $\xi = (v^p - u^p)_+$ , then

$$(2.13) \quad \int_{\Omega} \left( \frac{f(u)}{u^{p-1}} - \frac{f(v)}{v^{p-1}} \right) \xi dx \leq \int_{\Omega} \xi \left( \frac{L_{s,p}u}{u^{p-1}} - \frac{L_{s,p}v}{v^{p-1}} \right) dx.$$

Let us analyze each term in the previous inequality.

Using the definition of  $\xi$  we obtain that  $\left(\frac{f(u)}{u^{p-1}} - \frac{f(v)}{v^{p-1}}\right)\xi \geq 0$ . On the other hand, we have

$$\begin{aligned} J \equiv \int_{\Omega} \xi \left( \frac{L_{s,p}u}{u^{p-1}} - \frac{L_{s,p}v}{v^{p-1}} \right) dx &= \frac{1}{2} \int \int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} \left( \frac{\xi(x)}{u^{p-1}(x)} - \frac{\xi(y)}{u^{p-1}(y)} \right) dx dy \\ &\quad - \frac{1}{2} \int \int_Q \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x - y|^{N+ps}} \left( \frac{\xi(x)}{v^{p-1}(x)} - \frac{\xi(y)}{v^{p-1}(y)} \right) dx dy, \end{aligned}$$

where  $Q = \mathbb{R}^N \times \mathbb{R}^N \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ .

Notice that

$$\begin{aligned} &|u(x) - u(y)|^{p-2}(u(x) - u(y)) \left( \frac{\xi(x)}{u^{p-1}(x)} - \frac{\xi(y)}{u^{p-1}(y)} \right) = \\ &|u(x) - u(y)|^{p-2}(u(x) - u(y)) \left( \frac{v^p(x)}{u^{p-1}(x)} - \frac{v^p(y)}{u^{p-1}(y)} \right) \\ &- |u(x) - u(y)|^p. \end{aligned}$$

In the same way, we obtain that

$$\begin{aligned} &|v(x) - v(y)|^{p-2}(v(x) - v(y)) \left( \frac{\xi(x)}{v^{p-1}(x)} - \frac{\xi(y)}{v^{p-1}(y)} \right) = \\ &- |v(x) - v(y)|^{p-2}(v(x) - v(y)) \left( \frac{u^p(x)}{v^{p-1}(x)} - \frac{u^p(y)}{v^{p-1}(y)} \right) \\ &+ |v(x) - v(y)|^p. \end{aligned}$$

Thus

$$\begin{aligned} J &= \int \int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} \left( \frac{v^p(x)}{u^{p-1}(x)} - \frac{v^p(y)}{u^{p-1}(y)} \right) \\ &+ \int \int_Q \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x - y|^{N+ps}} \left( \frac{u^p(x)}{v^{p-1}(x)} - \frac{u^p(y)}{v^{p-1}(y)} \right) \\ &- \int \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} - \int \int_Q \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \\ &= \frac{1}{2} \int_{\Omega} \frac{L_{p,s}(u)}{u^p} v^p dx + \frac{1}{2} \int_{\Omega} \frac{L_{p,s}(v)}{v^{p-1}} u^p dx - \int \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy - \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} dx dy. \end{aligned}$$

Now, using Picone's inequality, we conclude that  $J \leq 0$ . Thus

$$\left( \frac{f(u)}{u^{p-1}} - \frac{f(v)}{v^{p-1}} \right) \xi \equiv 0$$

and then  $\xi = 0$  which implies that  $u \leq v$  in  $\Omega$ .  $\square$

**Remark 2.5.** The comparison result holds if we replace  $f(s)$  by  $g(s, x)$  where  $g$  is continuous in  $s$  for a.e  $x \in \Omega$ ,  $\frac{g(s, x)}{s}$  is decreasing for  $s > 0$  and  $g(s, x) > 0$  in  $\Omega$  for all  $s > 0$  fixed.

In the sequel we need the next results.

**Lemma 2.6.** Fix  $0 < \beta < \frac{N-ps}{2}$  and let  $w(x) = |x|^{-\gamma}$  with  $0 < \gamma < \frac{N-ps-2\beta}{p-1}$ , then there exists a positive constant  $\Lambda(\gamma) > 0$  such that

$$(2.14) \quad L_{s,p,\beta}(w) = \Lambda(\gamma) \frac{w^{p-1}}{|x|^{ps+2\beta}} \quad \text{a.e in } \mathbb{R}^N \setminus \{0\}.$$

*Proof.* We set  $r = |x|$  and  $\rho = |y|$ , then  $x = rx'$ ,  $y = \rho y'$  where  $|x'| = |y'| = 1$ . Thus

$$L_{s,p,\beta}(w) = \frac{1}{|x|^\beta} \int_0^{+\infty} |r^{-\gamma} - \rho^{-\gamma}|^{p-2} \frac{(r^{-\gamma} - \rho^{-\gamma})\rho^{N-1}}{\rho^\beta r^{N+ps}} \left( \int_{|y'|=1} \frac{dH^{n-1}(y')}{|x' - \frac{\rho}{r}y'|^{N+ps}} \right) d\rho.$$

We set  $\sigma = \frac{\rho}{r}$ , then

$$L_{s,p,\beta}(w) = \frac{w^{p-1}(x)}{|x|^{ps+2\beta}} \int_0^{+\infty} |1 - \sigma^{-\gamma}|^{p-2} (1 - \sigma^{-\gamma}) \sigma^{N-\beta-1} \left( \int_{|y'|=1} \frac{dH^{n-1}(y')}{|x' - \sigma y'|^{N+ps}} \right) d\sigma.$$

Defining

$$K(\sigma) = \int_{|y'|=1} \frac{dH^{n-1}(y')}{|x' - \sigma y'|^{N+ps}},$$

as in [17], we obtain that

$$(2.15) \quad K(\sigma) = 2 \frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \int_0^\pi \frac{\sin^{N-2}(\theta)}{(1 - 2\sigma \cos(\theta) + \sigma^2)^{\frac{N+ps}{2}}} d\theta.$$

Hence we conclude that

$$L_{s,p,\beta}(w) = \frac{w^{p-1}(x)}{|x|^{ps+2\beta}} \int_0^{+\infty} \psi(\sigma) d\sigma,$$

with

$$(2.16) \quad \psi(\sigma) = |1 - \sigma^{-\gamma}|^{p-2} (1 - \sigma^{-\gamma}) \sigma^{N-\beta-1} K(\sigma).$$

Define  $\Lambda(\gamma) \equiv \int_0^{+\infty} \psi(\sigma) d\sigma$ , to conclude we have just to show that  $0 < \Lambda(\gamma) < \infty$ .

We have

$$\Lambda(\gamma) = \int_0^1 \psi(\sigma) d\sigma + \int_1^\infty \psi(\sigma) d\sigma = I_1 + I_2.$$

Notice that  $K(\frac{1}{\xi}) = \xi^{N+ps} K(\xi)$  for any  $\xi > 0$ , then using the change of variable  $\xi = \frac{1}{\sigma}$  in  $I_1$ , there results that

$$(2.17) \quad \Lambda(\gamma) = \int_1^{+\infty} K(\sigma) (\sigma^\gamma - 1)^{p-1} \left( \sigma^{N-1-\beta-\gamma(p-1)} - \sigma^{\beta+ps-1} \right) d\sigma.$$

As  $\sigma \rightarrow \infty$ , we have

$$K(\sigma) (\sigma^\gamma - 1)^{p-1} \left( \sigma^{N-1-\beta-\gamma(p-1)} - \sigma^{\beta+ps-1} \right) \simeq \sigma^{-1-\beta-ps} \in L^1((2, \infty)).$$

Now, as,  $\sigma \rightarrow 1$ , we have

$$K(\sigma) (\sigma^\gamma - 1)^{p-1} \left( \sigma^{N-1-\beta-\gamma(p-1)} - \sigma^{\beta+ps-1} \right) \simeq (\sigma - 1)^{p-1-ps} \in L^1((1, 2)).$$

Therefore, combining the above estimates, we get  $|\Lambda(\gamma)| < \infty$ . Now, using the fact that  $0 < \gamma < \frac{N-ps-2\beta}{p-1}$ , then from (2.17) we reach that  $\Lambda(\gamma) > 0$ .

As a conclusion, we have proved that

$$L_{s,p,\beta}(w) = \Lambda(\gamma) \frac{w^{p-1}}{|x|^{ps+2\beta}} \text{ a.e. in } \mathbb{R}^N \setminus \{0\}.$$



Hence the result follows.  $\square$

As a consequence we have the following weighted Hardy inequality.

**Theorem 2.7.** *Let  $\beta < \frac{N-ps}{2}$ , then for all  $u \in C_0^\infty(\mathbb{R}^N)$ , we have*

$$(2.18) \quad 2\Lambda(\gamma) \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps+2\beta}} dx \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta},$$

where  $\Lambda(\gamma)$  is defined in (2.17).

*Proof.* Let  $u \in C_0^\infty(\mathbb{R}^N)$  and  $w(x) = |x|^{-\gamma}$  with  $\gamma < \frac{N-ps-2\beta}{p-1}$ . By Lemma 2.6, we have

$$L_{p,s,\beta}(w) = \Lambda(\gamma) \frac{w^{p-1}}{|x|^{ps+2\beta}}.$$

It is clear that  $\frac{w^{p-1}}{|x|^{ps+2\beta}} \in L_{loc}^1(\mathbb{R}^N)$ . Thus using Picone inequality in Lemma 2.3, it follows that

$$\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} \geq \langle L_{p,s,\beta} w, \frac{|u|^p}{w^{p-1}} \rangle = \Lambda(\gamma) \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps+2\beta}} dx.$$

Thus we conclude.  $\square$

**Remark 2.8.** Let analyze the behavior of the constant  $\Lambda(\gamma)$  in inequality (2.18). Recall that, for  $\gamma < \frac{N-ps-2\beta}{p-1}$ ,

$$\Lambda(\gamma) = \int_1^{+\infty} K(\sigma) (\sigma^\gamma - 1)^{p-1} \left( \sigma^{N-1-\beta-\gamma(p-1)} - \sigma^{\beta+ps-1} \right) d\sigma,$$

then

$$\Lambda'(\gamma) = (p-1) \int_1^{+\infty} K(\sigma) \log(\sigma) (\sigma^\gamma - 1)^{p-2} \left( \sigma^{N-1-\beta-\gamma(p-1)} - \sigma^{\beta+ps+\gamma-1} \right) d\sigma.$$

It is clear that if  $\gamma_0 = \frac{N-ps-2\beta}{p}$ , then  $\Lambda'(\gamma_0) = 0$ ,  $\Lambda'(\gamma) > 0$  if  $\gamma < \gamma_0$  and  $\Lambda'(\gamma) < 0$  if  $\gamma > \gamma_0$ . Thus

$$\max_{\{0 < \gamma < \frac{N-ps-2\beta}{p-1}\}} \Lambda(\gamma) = \Lambda(\gamma_0).$$

Hence

$$(2.19) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} \geq 2\Lambda(\gamma_0) \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps+2\beta}} dx.$$

Notice that for  $\beta = 0$ , then  $2\Lambda(\gamma_0) = 2\Lambda(\frac{N-ps}{p}) \equiv \Lambda_{N,p,s}$  given in (1.3). Therefore, we have the next optimality result.

**Theorem 2.9.** *Define*

$$\Lambda_{N,p,s,\gamma} = \inf_{\{\phi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}\}} \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{N+ps} |x|^\beta |y|^\beta} dx dy}{\int_{\mathbb{R}^N} \frac{|\phi(x)|^p}{|x|^{ps+2\beta}} dx},$$

then  $\Lambda_{N,p,s,\gamma} = 2\Lambda(\gamma_0)$ .

*Proof.* From (2.19), it follows that  $\Lambda_{N,p,s,\gamma} \geq 2\Lambda(\gamma_0)$ , hence to conclude we have just to prove the reverse inequality.

We closely follow the argument used in [16].

Let  $w_0(x) = |x|^{-\gamma_0}$ , by Lemma 2.6, we have

$$L_{p,s,\beta}(w_0) = \Lambda(\gamma_0) \frac{w_0^{p-1}}{|x|^{ps+2\beta}}.$$

We set

$$M_n = \{x \in \mathbb{R}^N : 1 \leq |x| < n\} \text{ and } O_n = \{x \in \mathbb{R}^N : |x| \geq n\}.$$

and define

$$w_n = \begin{cases} 1 - n^{-\gamma_0} & \text{if } x \in B_1(0), \\ |x|^{-\gamma_0} - n^{-\gamma_0} & \text{if } x \in M_n, \\ 0 & \text{if } x \in O_n. \end{cases}$$

By a direct computation, we get easily that  $w_n \in X_0^{s,p,\beta}(\mathbb{R}^N)$ .

Hence

$$\langle L_{p,s,\beta}(w_0), w_n \rangle = \Lambda(\gamma_0) \int_{\mathbb{R}^N} \frac{w_n w_0^{p-1}}{|x|^{ps+2\beta}}.$$

Thus

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(w_n(x) - w_n(y))|w_0(x) - w_0(y)|^{p-2}(w(x) - w(y))}{|x - y|^{N+ps}|x|^\beta|y|^\beta} dx dy = 2\Lambda(\gamma_0) \int_{\mathbb{R}^N} \frac{w_n w_0^{p-1}}{|x|^{ps+2\beta}}.$$

Let analyze each term in the previous identity. As in [16] we obtain that

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(w_n(x) - w_n(y))|w_0(x) - w_0(y)|^{p-2}(w(x) - w(y))}{|x - y|^{N+ps}|x|^\beta|y|^\beta} dx dy \geq \\ & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^p}{|x - y|^{N+ps}|x|^\beta|y|^\beta} dx dy. \end{aligned}$$

On the other hand we have

$$\int_{\mathbb{R}^N} \frac{w_n w_0^{p-1}}{|x|^{ps+2\beta}} dx = \int_{\mathbb{R}^N} \frac{w_n^p}{|x|^{ps+2\beta}} dx + I_n + J_n,$$

where

$$I_n = \int_{B_1(0)} (1 - n^{-\gamma_0})(w_0^{p-1} - (1 - n^{-\gamma_0})^{p-1}) \frac{dx}{|x|^{ps+\beta}},$$

and

$$J_n = \int_{M_n} (w_0(x) - n^{-\gamma_0})(w_0^{p-1} - (w_0(x) - n^{-\gamma_0})^{p-1}) \frac{dx}{|x|^{ps+\beta}}.$$

It is clear that  $I_n, J_n \geq 0$ , using a direct computation we can prove that

$$I_n + J_n \leq C \text{ for all } n \geq 1.$$

Thus, combining the above estimates, we reach that

$$(2.20) \quad \Lambda_{N,p,s,\gamma} \leq \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^p}{|x - y|^{N+ps}|x|^\beta|y|^\beta} dx dy}{\int_{\mathbb{R}^N} \frac{|w_n(x)|^p}{|x|^{ps+\beta}} dx}$$

$$(2.21) \quad \leq 2\Lambda(\gamma_0) \left( 1 + \frac{I_n + J_n}{\int_{\mathbb{R}^N} \frac{|w_n(x)|^p}{|x|^{ps+\beta}} dx} \right).$$

Since  $\int_{\mathbb{R}^N} \frac{|w_n(x)|^p}{|x|^{ps+\beta}} dx \uparrow \infty$  as  $n \rightarrow \infty$ , then passing to the limit in (2.20), it follows that

$$\Lambda_{N,p,s,\gamma} \leq 2\Lambda(\gamma_0)$$

and then the result follows.  $\square$

In the sequel we need to use a version of the Hardy inequality in bounded domains. More precisely, we have the next result.

**Lemma 2.10.** *Let  $\Omega$  be a bounded regular domain such that  $0 \in \Omega$ , then there exists a constant  $C \equiv C(\Omega, s, p, N) > 0$  such that for all  $u \in C_0^\infty(\Omega)$ , we have*

$$(2.22) \quad C \int_{\Omega} \frac{|u(x)|^p}{|x|^{ps+2\beta}} dx \leq \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta}.$$

*Proof.* Fix  $u \in C_0^\infty(\Omega)$  and let  $\tilde{u}$ , be the extension of  $u$  to  $\mathbb{R}^N$  defined in Lemma 2.1. Then from Theorem 2.7, we get

$$2\Lambda(\gamma) \int_{\mathbb{R}^N} \frac{|\tilde{u}(x)|^p}{|x|^{ps+2\beta}} dx \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} \leq \|\tilde{u}\|_{X^{s,p,\beta}(\mathbb{R}^N)}^p \leq C \|u\|_{X^{s,p,\beta}(\Omega)}^p.$$

Since  $\tilde{u}|_{\Omega} = u$ , then from Remark 2.2 we conclude that

$$\begin{aligned} 2\Lambda(\gamma) \int_{\Omega} \frac{|u(x)|^p}{|x|^{ps+2\beta}} dx &\leq C \|u\|_{X^{s,p,\beta}(\Omega)}^p \\ &\leq C_1 \|u\|_{X_0^{s,p,\beta}(\Omega)}^p = C_1 \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta}. \end{aligned}$$

Hence we reach the desired result.  $\square$

Now, we are able to proof Theorem 1.3.

**Proof of Theorem 1.3.**

We follow closely the arguments used in [2]. Let  $u \in C_0^\infty(\Omega)$  and define  $\alpha = \frac{N-ps}{p}$ , then  $w(x) = |x|^{-\alpha}$  and  $v(x) = \frac{u(x)}{w(x)}$ .

Recall that from the result of [16], we have

$$(2.23) \quad h_s(u) \geq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^{\frac{N-ps}{2}}} \frac{dy}{|y|^{\frac{N-ps}{2}}},$$

hence we will analyze the right hand side of the previous inequality.

Notice that

$$\begin{aligned} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} w(x)^{\frac{p}{2}} w(y)^{\frac{p}{2}} &= \frac{|w(y)u(x) - w(x)u(y)|^p}{|x - y|^{N+ps}} \frac{1}{(w(x)w(y))^{\frac{p}{2}}} \\ &= \frac{\left| (u(x) - u(y)) - \frac{u(y)}{w(y)}(w(x) - w(y)) \right|^p}{|x - y|^{N+ps}} \left( \frac{w(y)}{w(x)} \right)^{\frac{p}{2}} = f_1(x, y). \end{aligned}$$

In the same way, thanks to the symmetry of  $f_1(x, y)$ , it immediately follows that

$$\frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} (w(x))^{\frac{p}{2}} (w(y))^{\frac{p}{2}} = \frac{\left| (u(y) - u(x)) - \frac{u(x)}{w(x)} (w(y) - w(x)) \right|^p}{|x - y|^{N+ps}} \left( \frac{w(x)}{w(y)} \right)^{\frac{p}{2}} = f_2(x, y).$$

Hence,

$$h_s(u) \geq \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_1(x, y) dx dy + \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_2(x, y) dx dy.$$

Since  $f_1$  and  $f_2$  are positive functions, it follows that

$$h_s(u) \geq \frac{1}{2} \int_{\Omega} \int_{\Omega} f_1(x, y) dx dy + \frac{1}{2} \int_{\Omega} \int_{\Omega} f_2(x, y) dx dy.$$

Using the fact that  $\Omega$  is a bounded domain, we obtain that for all  $(x, y) \in (\Omega \times \Omega)$  and  $q < p$ ,

$$\frac{1}{|x - y|^{N+ps}} \geq \frac{C(\Omega)}{|x - y|^{N+qs}}$$

and

$$Q(x, y) \equiv \frac{(w(x)w(y))^{\frac{p}{2}}}{w(x)^p + w(y)^p} \leq C.$$

Define

$$D(x, y) \equiv \left( \frac{w(x)}{w(y)} \right)^{\frac{p}{2}} + \left( \frac{w(y)}{w(x)} \right)^{\frac{p}{2}} \equiv \frac{w(x)^p + w(y)^p}{(w(x)w(y))^{\frac{p}{2}}},$$

then  $Q(x, y)D(x, y) = 1$ . Thus

$$\begin{aligned} f_1(x, y) &\geq C(\Omega)Q(x, y) \left( \frac{w(y)}{w(x)} \right)^{\frac{p}{2}} \times \\ &\left[ \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}} - p \frac{|u(x) - u(y)|^{p-2}}{|x - y|^{N+qs}} \left\langle u(x) - u(y), \frac{u(y)}{w(y)} (w(x) - w(y)) \right\rangle \right. \\ &\quad \left. + C(p) \frac{\left| \frac{u(y)}{w(y)} (w(x) - w(y)) \right|^p}{|x - y|^{N+qs}} \right]. \end{aligned}$$

Hence

$$\begin{aligned} f_1(x, y) &\geq \left[ C(\Omega)Q(x, y) \left( \frac{w(y)}{w(x)} \right)^{\frac{p}{2}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}} \right] \\ &- \left[ pC(\Omega)Q(x, y) \left( \frac{w(y)}{w(x)} \right)^{\frac{p}{2}} \frac{|u(x) - u(y)|^{p-1}}{|x - y|^{N+qs}} \left| \frac{u(y)}{w(y)} \right| |w(x) - w(y)| \right]. \end{aligned}$$

In the same way we reach that

$$\begin{aligned} f_2(x, y) &\geq \left[ C(\Omega)Q(x, y) \left( \frac{w(x)}{w(y)} \right)^{\frac{p}{2}} \frac{|u(y) - u(x)|^p}{|x - y|^{N+qs}} \right] \\ &- \left[ pC(\Omega)Q(x, y) \left( \frac{w(x)}{w(y)} \right)^{\frac{p}{2}} \frac{|u(x) - u(y)|^{p-1}}{|x - y|^{N+qs}} \left| \frac{u(x)}{w(x)} \right| |w(x) - w(y)| \right]. \end{aligned}$$

Therefore,

$$\begin{aligned}
h_s(u) &\geq C(\Omega) \int_{\Omega} \int_{\Omega} Q(x, y) \left( \left( \frac{w(y)}{w(x)} \right)^{\frac{p}{2}} + \left( \frac{w(x)}{w(y)} \right)^{\frac{p}{2}} \right) \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}} dx dy \\
&- pC(\Omega) \int_{\Omega} \int_{\Omega} \left[ Q(x, y) \left( \frac{w(y)}{w(x)} \right)^{\frac{p}{2}} \frac{|u(x) - u(y)|^{p-1}}{|x - y|^{N+qs}} \left| \frac{u(y)}{w(y)} \right| |(w(x) - w(y))| \right] dx dy \\
&- pC(\Omega) \int_{\Omega} \int_{\Omega} \left[ Q(x, y) \left( \frac{w(x)}{w(y)} \right)^{\frac{p}{2}} \frac{|u(x) - u(y)|^{p-1}}{|x - y|^{N+qs}} \left| \frac{u(x)}{w(x)} \right| |(w(x) - w(y))| \right] dx dy.
\end{aligned}$$

Thus

$$\begin{aligned}
(2.24) \quad h_s(u) &\geq C(\Omega) \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}} dx dy \\
&- C_1(\Omega, p) \int_{\Omega} \int_{\Omega} (h_1(x, y) + h_2(x, y)) dx dy,
\end{aligned}$$

with

$$\begin{aligned}
h_1(x, y) &= Q(x, y) \left( \frac{w(y)}{w(x)} \right)^{\frac{p}{2}} \frac{|u(x) - u(y)|^{p-1}}{|x - y|^{N+qs}} \left| \frac{u(y)}{w(y)} \right| |(w(x) - w(y))|, \\
h_2(x, y) &= Q(x, y) \left( \frac{w(x)}{w(y)} \right)^{\frac{p}{2}} \frac{|u(x) - u(y)|^{p-1}}{|x - y|^{N+qs}} \left| \frac{u(x)}{w(x)} \right| |(w(x) - w(y))|.
\end{aligned}$$

Since  $h_1(x, y)$  and  $h_2(x, y)$  are symmetric functions, we just have to estimate  $\int_{\Omega} \int_{\Omega} h_2(x, y) dx dy$ .

Using Young inequality, we get

$$\begin{aligned}
(2.25) \quad \int_{\Omega} \int_{\Omega} h_2(x, y) dx dy &\leq \varepsilon \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}} dx dy \\
&+ C(\varepsilon) \int_{\Omega} \int_{\Omega} G(x, y) dx dy,
\end{aligned}$$

with

$$G(x, y) = (Q(x, y))^p \left( \frac{w(x)}{w(y)} \right)^{\frac{p^2}{2}} \left| \frac{u(x)}{w(x)} \right|^p \frac{|(w(x) - w(y))|^p}{|x - y|^{N+qs}}.$$

We claim that

$$I \equiv \int_{\Omega} \int_{\Omega} G(x, y) dx dy \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^{\frac{N-ps}{2}}} \frac{dy}{|y|^{\frac{N-ps}{2}}}.$$

Notice that

$$I = \int_{\Omega} \int_{\Omega} \frac{(u(x))^p}{|x - y|^{N+qs}} \frac{(w(x))^{p^2-p} |(w(x) - w(y))|^p}{(w(x)^p + w(y)^p)^p} dx dy,$$

then

$$I = \int_{\Omega} u^p(x) \left[ \int_{\Omega} \frac{||x|^{\alpha} - |y|^{\alpha}|^p}{(|x|^{\alpha p} + |y|^{\alpha p})^p} \frac{|y|^{\alpha p(p-1)}}{|x - y|^{N+qs}} dy \right] dx.$$

To compute the above integral, we closely follow the arguments used in [17]. We set  $y = \rho y'$  and  $x = rx'$  with  $|x'| = |y'| = 1$ , then taking in consideration that  $\Omega \subset B_0(R)$ , it follows that

$$\begin{aligned} I &= \int_{\Omega} u^p(x) \left[ \int_{\Omega} \frac{||x|^{\alpha} - |y|^{\alpha}|^p}{(|x|^{\alpha p} + |x|^{\alpha p})^p} \frac{|y|^{\alpha p(p-1)}}{|x - y|^{N+qs}} dy \right] dx \\ &\leq \int_{\Omega} u^p(x) \int_0^R \frac{(|r^{\alpha} - \rho^{\alpha}|^p \rho^{\alpha p(p-1)+N-1})}{(r^{p\alpha} + \rho^{p\alpha})^p} \left( \int_{\mathbb{S}^{N-1}} \frac{dy'}{|\rho y' - rx'|^{N+qs}} \right) d\rho dx. \end{aligned}$$

We set  $\rho = r\sigma$ , then

$$\begin{aligned} I &\leq \int_{\Omega} \frac{u^p(x)}{|x|^{qs}} \int_0^{\frac{R}{r}} \frac{|1 - \sigma^{\alpha}|^p \sigma^{\alpha p(p-1)+N-1}}{(1 + \sigma^{\alpha p})^p} \left( \int_{\mathbb{S}^{N-1}} \frac{dy'}{|\sigma y' - x'|^{N+qs}} \right) d\sigma dx \\ &= \int_{\Omega} \frac{u^p(x)}{|x|^{qs}} \int_0^{\frac{R}{r}} \frac{|1 - \sigma^{\alpha}|^p \sigma^{\alpha p(p-1)+N-1}}{(1 + \sigma^{\alpha p})^p} K(\sigma) d\sigma dx \leq \mu \int_{\Omega} \frac{u^p(x)}{|x|^{qs}} dx, \end{aligned}$$

where

$$\mu = \int_0^{\infty} \frac{|1 - \sigma^{\alpha}|^p \sigma^{\alpha p(p-1)+N-1}}{(1 + \sigma^{\alpha p})^p} K(\sigma) d\sigma$$

and

$$K(\sigma) = 2 \frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \int_0^{\pi} \frac{\sin^{N-2}(\theta)}{(1 - 2\sigma \cos(\theta) + \sigma^2)^{\frac{N+qs}{2}}} d\theta.$$

Let us show that  $\mu < \infty$ .

It is clear that, as  $\sigma \rightarrow \infty$ , we have

$$\frac{(|1 - \sigma^{\alpha}|^p \sigma^{\alpha p(p-1)+N-1})}{(1 + \sigma^{\alpha p})^p} K(\sigma) \preceq \sigma^{-1-qs} \in L^1((1, \infty)).$$

Now, taking in consideration that  $K(\sigma) \leq C|1 - \sigma|^{-1-ps}$  as  $s \rightarrow 1$ , and following the same computation as in Lemma 2.6, it follows that

$$\int_0^1 \frac{(1 - \sigma^{\alpha})^p \sigma^{\alpha p(p-1)+N-1}}{(1 + \sigma^{\alpha p})^p} K(\sigma) d\sigma < \infty.$$

Thus  $\mu < \infty$ .

Hence combining the above estimates, there results that

$$I \leq C \int_{\Omega} \frac{u^p(x)}{|x|^{qs}} dx.$$

Since  $u(x) = v(x)|x|^{-\frac{N-ps}{p}}$ , then

$$I \leq C \int_{\Omega} \frac{|v(x)|^p}{|x|^{N-s(p-q)}} dx.$$

Let  $\beta_0 = \frac{N-ps}{2} + \frac{(q-p)s}{2}$ , then  $\beta_0 < \frac{N-ps}{2}$ . Applying Lemma 2.10, we obtain that

$$\begin{aligned} I &\leq C(\Omega) \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps} |x|^{\beta_0} |y|^{\beta_0}} dy dx \\ &\leq C_1(\Omega) \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps} |x|^{\frac{N-ps}{2}} |y|^{\frac{N-ps}{2}}} dy dx \\ &\leq C_1(\Omega) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps} |x|^{\frac{N-ps}{2}} |y|^{\frac{N-ps}{2}}} dy dx. \end{aligned}$$

Therefore, using again estimate (2.23), we reach that

$$I \leq C_2(\Omega) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^{\frac{N-ps}{2}}} \frac{dy}{|y|^{\frac{N-ps}{2}}}$$

and the claim follows.

As a direct consequence of the above estimates, we have proved that

$$(2.26) \quad \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}} dx dy \leq C_3 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^{\frac{N-ps}{2}}} \frac{dy}{|y|^{\frac{N-ps}{2}}}.$$

Thus

$$\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}} dx dy \leq Ch_s(u),$$

and the result follows at once. ■

We are now in position to prove the Theorem 1.4.

**Proof of Theorem 1.4.** Recall that  $\alpha = \frac{N-ps}{p}$ . Since  $\alpha p_{s,q}^* = \frac{N(N-ps)}{N-qs} < N$ , it follows that

$$\int_{\Omega} \frac{|u(x)|^{p_{s,q}^*}}{|x|^{\alpha p_{s,q}^*}} dx < \infty, \text{ for all } u \in \mathcal{C}_0^\infty(\mathbb{R}^N).$$

To prove (1.7), we will use estimate (2.26) and the fractional Sobolev inequality.

Fix  $u \in \mathcal{C}_0^\infty(\Omega)$  and define  $u_1(x) = \frac{u(x)}{|x|^\alpha}$ . By (2.26), we obtain that

$$C(\Omega) \int_{\Omega} \int_{\Omega} \frac{|u_1(x) - u_1(y)|^p}{|x - y|^{N+qs}} dx dy \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^{\frac{N-ps}{2}}} \frac{dy}{|y|^{\frac{N-ps}{2}}}.$$

Now, using Sobolev inequality, there results that

$$S \left( \int_{\Omega} |u_1(x)|^{p_{s,q}^*} dx \right)^{\frac{p}{p_{s,q}^*}} \leq \int_{\Omega} \int_{\Omega} \frac{|u_1(x) - u_1(y)|^p}{|x - y|^{N+qs}} dx dy,$$

where  $p_{s,q}^* = \frac{pN}{N-qs}$ . Hence, substituting  $u_1$  by its value, we get

$$(2.27) \quad \left( \int_{\Omega} \frac{|u(x)|^{p_{s,q}^*}}{|x|^{\alpha p_{s,q}^*}} dx \right)^{\frac{p}{p_{s,q}^*}} \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\alpha} \frac{dy}{|y|^\alpha}.$$

If we set  $\beta = \frac{N-ps}{2}$ , then inequality (2.27) can be written in the form

$$(2.28) \quad \left( \int_{\Omega} \frac{|u(x)|^{p_{s,q}^*}}{|x|^{2\beta \frac{p_{s,q}^*}{p}}} dx \right)^{\frac{p}{p_{s,q}^*}} \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta}.$$

■

As a consequence, we will prove the fractional Caffarelli-Kohn-Nirenberg inequality given in Theorem 1.5.

**Proof of Theorem 1.5.** Let  $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ , without loss of generality, we can assume that  $u \geq 0$ .

Using the fact that  $\beta < \frac{N-ps}{2}$ , we easily get that  $\int_{\mathbb{R}^N} \frac{|u(x)|^{p_s^*}}{|x|^{2\beta \frac{p_s^*}{p}}} dx < \infty$ .

From now and for simplicity of typing, we denote by  $C, C_1, C_2, \dots$  any universal constant that does not depend on  $u$  and can change from a line to another.

We set  $\tilde{u}(x) = \frac{u(x)}{w_1(x)}$ , where  $w_1(x) = |x|^{\frac{2\beta}{p}}$ , then

$$(2.29) \quad \left( \int_{\mathbb{R}^N} \frac{|u(x)|^{p_s^*}}{|x|^{2\beta \frac{p_s^*}{p}}} dx \right)^{\frac{p}{p_s^*}} = \left( \int_{\mathbb{R}^N} |\tilde{u}|^{p_s^*} dx \right)^{\frac{p}{p_s^*}}.$$

Using Sobolev inequality, it follows that

$$(2.30) \quad S \left( \int_{\mathbb{R}^N} |\tilde{u}|^{p_s^*} dx \right)^{\frac{p}{p_s^*}} \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x - y|^{N+ps}} dx dy.$$

To get the desired result we just have to show that

$$(2.31) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x - y|^{N+ps}} dx dy \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta}$$

for some positive constant  $C$ .

Using the definition of  $\tilde{u}$ , we get

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_1(x)\tilde{u}(x) - w_1(y)\tilde{u}(y)|^p}{|x - y|^{N+ps}} \frac{dx}{w_1^{\frac{p}{2}}(x)} \frac{dy}{w_1^{\frac{p}{2}}(y)}.$$

Notice that

$$\begin{aligned} & \frac{|w_1(x)\tilde{u}(x) - w_1(y)\tilde{u}(y)|^p}{|x - y|^{N+ps}} \frac{1}{w_1^{\frac{p}{2}}(x)} \frac{1}{w_1^{\frac{p}{2}}(y)} = \\ & \frac{\left| (\tilde{u}(x) - \tilde{u}(y)) - w_1(y)\tilde{u}(y) \left( \frac{1}{w_1(x)} - \frac{1}{w_1(y)} \right) \right|^p}{|x - y|^{N+ps}} \left( \frac{w_1(x)}{w_1(y)} \right)^{\frac{p}{2}} \equiv \tilde{f}_1(x, y). \end{aligned}$$

In the same way we have

$$\begin{aligned} & \frac{|w_1(x)\tilde{u}(x) - w_1(y)\tilde{u}(y)|^p}{|x - y|^{N+ps}} \frac{1}{w_1^{\frac{p}{2}}(x)} \frac{1}{w_1^{\frac{p}{2}}(y)} = \\ & \frac{\left| (\tilde{u}(y) - \tilde{u}(x)) - w_1(x)\tilde{u}(x) \left( \frac{1}{w_1(y)} - \frac{1}{w_1(x)} \right) \right|^p}{|x - y|^{N+ps}} \left( \frac{w_1(y)}{w_1(x)} \right)^{\frac{p}{2}} \equiv \tilde{f}_2(x, y). \end{aligned}$$

Since

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \tilde{f}_1(x, y) dx dy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \tilde{f}_2(x, y) dx dy,$$

we get

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \tilde{f}_1(x, y) dx dy + \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \tilde{f}_2(x, y) dx dy.$$



Notice that

$$\begin{aligned} \tilde{f}_1(x, y) &\geq \left( \frac{w_1(x)}{w_1(y)} \right)^{\frac{p}{2}} \times \\ &\left[ \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x - y|^{N+ps}} - p \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p-2}}{|x - y|^{N+ps}} \langle \tilde{u}(x) - \tilde{u}(y), w_1(y) \tilde{u}(y) \left( \frac{1}{w_1(x)} - \frac{1}{w_1(y)} \right) \rangle \right. \\ &\quad \left. + C(p) \frac{|w_1(y) \tilde{u}(y) \left( \frac{1}{w_1(x)} - \frac{1}{w_1(y)} \right)|^p}{|x - y|^{N+ps}} \right], \end{aligned}$$

Hence

$$\begin{aligned} \tilde{f}_1(x, y) &\geq \left( \frac{w_1(x)}{w_1(y)} \right)^{\frac{p}{2}} \times \\ &\left[ \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x - y|^{N+ps}} - p \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p-1}}{|x - y|^{N+ps}} |w_1(y) \tilde{u}(y) \left( \frac{1}{w_1(x)} - \frac{1}{w_1(y)} \right)| \right]. \end{aligned}$$

Using Young inequality, we get the existence of  $C_1, C_2 > 0$  such that

$$\begin{aligned} \tilde{f}_1(x, y) &\geq \left( \frac{w_1(x)}{w_1(y)} \right)^{\frac{p}{2}} \times \\ &\left[ C_1 \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x - y|^{N+ps}} - C_2 \frac{|w_1(y) \tilde{u}(y) \left( \frac{1}{w_1(x)} - \frac{1}{w_1(y)} \right)|^p}{|x - y|^{N+ps}} \right]. \end{aligned}$$

In the same way and using that  $\tilde{f}_1, \tilde{f}_2$  are symmetric functions, it follows that

$$\begin{aligned} f_2(x, y) &\geq \left( \frac{w_1(y)}{w_1(x)} \right)^{\frac{p}{2}} \times \\ &\left[ C_1 \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x - y|^{N+ps}} - C_2 \frac{|w_1(x) \tilde{u}(x) \left( \frac{1}{w_1(y)} - \frac{1}{w_1(x)} \right)|^p}{|x - y|^{N+ps}} \right]. \end{aligned}$$

Thus we get the existence of positive constants  $C_1, C_2, C_3$  such that

$$\begin{aligned} &\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} \geq \\ &C_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x - y|^{N+ps}} \left[ \left( \frac{w_1(y)}{w_1(x)} \right)^{\frac{p}{2}} + \left( \frac{w_1(x)}{w_1(y)} \right)^{\frac{p}{2}} \right] dx dy \\ &- C_2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( \frac{w_1(x)}{w_1(y)} \right)^{\frac{p}{2}} \frac{|w_1(y) \tilde{u}(y) \left( \frac{1}{w_1(x)} - \frac{1}{w_1(y)} \right)|^p}{|x - y|^{N+qs}} dx dy \\ &- C_3 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( \frac{w_1(y)}{w_1(x)} \right)^{\frac{p}{2}} \frac{|w_1(x) \tilde{u}(x) \left( \frac{1}{w_1(y)} - \frac{1}{w_1(x)} \right)|^p}{|x - y|^{N+qs}} dx dy. \end{aligned}$$

Since

$$\left[ \left( \frac{w_1(y)}{w_1(x)} \right)^{\frac{p}{2}} + \left( \frac{w_1(x)}{w_1(y)} \right)^{\frac{p}{2}} \right] \geq 1,$$

it follows that

$$\begin{aligned}
 (2.32) \quad & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}(x) - \tilde{u}(y)|^p}{|x - y|^{N+ps}} dx dy \leq C_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} \\
 & + C_2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( \frac{w_1(x)}{w_1(y)} \right)^{\frac{p}{2}} \frac{|w_1(y)\tilde{u}(y)(\frac{1}{w_1(x)} - \frac{1}{w_1(y)})|^p}{|x - y|^{N+qs}} dx dy \\
 & + C_3 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( \frac{w_1(y)}{w_1(x)} \right)^{\frac{p}{2}} \frac{|w_1(x)\tilde{u}(x)(\frac{1}{w_1(y)} - \frac{1}{w_1(x)})|^p}{|x - y|^{N+qs}} dx dy.
 \end{aligned}$$

We set

$$g_1(x, y) = \left( \frac{w_1(y)}{w_1(x)} \right)^{\frac{p}{2}} \frac{|w_1(x)\tilde{u}(x)(\frac{1}{w_1(y)} - \frac{1}{w_1(x)})|^p}{|x - y|^{N+ps}}$$

and

$$g_2(x, y) = \left( \frac{w_1(x)}{w_1(y)} \right)^{\frac{p}{2}} \frac{|w_1(y)\tilde{u}(y)(\frac{1}{w_1(x)} - \frac{1}{w_1(y)})|^p}{|x - y|^{N+ps}}.$$

It is clear that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g_1(x, y) dx dy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g_2(x, y) dx dy,$$

therefore, to get the desired result, we just have to show that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g_1(x, y) dx dy \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta}.$$

Going back to the definition of  $\tilde{u}$  and  $w_1$ , we reach that

$$g_1(x, y) = \frac{|u(x)|^p \left| |x|^{\frac{2\beta}{p}} - |y|^{\frac{2\beta}{p}} \right|^p}{|x|^{3\beta} |y|^\beta |x - y|^{N+ps}}.$$

We closely follow the same type of computation as in the proof of Lemma 2.6.

We have

$$\begin{aligned}
 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g_1(x, y) dx dy &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^p \left| |x|^{\frac{2\beta}{p}} - |y|^{\frac{2\beta}{p}} \right|^p}{|x|^{3\beta} |y|^\beta |x - y|^{N+ps}} dx dy \\
 &= \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{3\beta}} \left( \int_{\mathbb{R}^N} \frac{\left| |x|^{\frac{2\beta}{p}} - |y|^{\frac{2\beta}{p}} \right|^p}{|y|^\beta |x - y|^{N+ps}} dy \right) dx.
 \end{aligned}$$

We set  $r = |x|$  and  $\rho = |y|$ , then  $x = rx'$ ,  $y = \rho y'$  with  $|x'| = |y'| = 1$ , then

$$\begin{aligned}
 & \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{3\beta}} \left( \int_{\mathbb{R}^N} \frac{\left| |x|^{\frac{2\beta}{p}} - |y|^{\frac{2\beta}{p}} \right|^p}{|y|^\beta |x - y|^{N+ps}} dy \right) dx = \\
 & \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{3\beta}} \left[ \int_0^{+\infty} \frac{r^{\frac{2\beta}{p}} - \rho^{\frac{2\beta}{p}}}{\rho^\beta} \rho^{N-1} \left( \int_{|y'|=1} \frac{dH^{n-1}(y')}{|rx' - \rho y'|^{N+ps}} \right) d\rho \right] dx.
 \end{aligned}$$

Let  $\sigma = \frac{\rho}{r}$ , then

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{3\beta}} \left( \int_{\mathbb{R}^N} \frac{\left| |x|^{\frac{2\beta}{p}} - |y|^{\frac{2\beta}{p}} \right|^p}{|y|^\beta |x-y|^{N+ps}} dy \right) dx = \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{2\beta+ps}} \left[ \int_0^{+\infty} |1 - \sigma^{\frac{2\beta}{p}}|^p \sigma^{N-1-\beta} K(\sigma) d\sigma \right] dx,$$

where  $K$  is defined in (2.15). Since

$$\int_0^{+\infty} |1 - \sigma^{\frac{2\beta}{p}}|^p \sigma^{N-1-\beta} K(\sigma) d\sigma \equiv C_3 < \infty,$$

it follows that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g_1(x, y) dx dy = C_3 \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{2\beta+ps}} dx.$$

Now, using inequality (2.18), we get

$$(2.33) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g_1(x, y) dx dy \leq C_4 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta}.$$

Combining (2.29), (2.30), (2.33) and (2.32), we reach the desired result.  $\blacksquare$

In the case where  $\Omega$  is a regular bounded domain containing the origin, we have the following version of Theorem 1.5.

**Theorem 2.11.** *Assume that  $\Omega$  is a regular bounded domain with  $0 \in \Omega$ , then there exists a positive constant  $C \equiv C(\Omega, N, p, s, \beta)$  such that for all  $\phi \in \mathcal{C}_0^\infty(\Omega)$ , we have*

$$(2.34) \quad \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} \geq C \left( \int_{\Omega} \frac{|\phi(x)|^{p_s^*}}{|x|^{2\beta \frac{p_s^*}{p}}} dx \right)^{\frac{p}{p_s^*}}.$$

*Proof.* Let  $\phi \in \mathcal{C}_0^\infty(\Omega)$  and define  $\tilde{\phi}$  to be the extension of  $\phi$  to  $\mathbb{R}^N$  given in Lemma 2.1, then using the fact that  $\Omega$  is a regular bounded domain, we reach that

$$\|\tilde{\phi}\|_{X^{s,p,\beta}(\mathbb{R}^N)} \leq C_1 \|\phi\|_{X^{s,p,\beta}(\Omega)} \leq C_1 \left( \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+ps}} \frac{dx dy}{|x|^\beta |y|^\beta} \right)^{\frac{1}{p}}.$$

Now, applying Theorem 1.5 to  $\tilde{\phi}$ , it follows that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{\phi}(x) - \tilde{\phi}(y)|^p}{|x-y|^{N+ps}} \frac{dx}{|x|^\beta} \frac{dy}{|y|^\beta} \geq S(\beta) \left( \int_{\mathbb{R}^N} \frac{|\tilde{\phi}(x)|^{p_s^*}}{|x|^{2\beta \frac{p_s^*}{p}}} dx \right)^{\frac{p}{p_s^*}}.$$

Hence combining the above estimates we get the desired result.  $\square$

**Remark 2.12.**

- (1) In the previous sections we have assumed that  $\beta > 0$ , however using the same arguments we can prove that the main Theorems 1.4 and 1.5 hold if  $\beta \in (-ps, 0]$ . This follows in particular using the fact that  $\Lambda_{N,p,s,\gamma}$  given in Theorem 2.9 is well defined if  $\beta > -ps$ .

- (2) If  $\beta < -ps$ , then  $\mathcal{C}_0^\infty(\mathbb{R}^N) \not\subseteq X^{s,p,\beta}(\mathbb{R}^N)$ . To see that we fix  $\phi \in \mathcal{C}_0^\infty(B_4(0))$  such that  $0 \leq \phi \leq 1$  and  $\phi = 1$  in  $B_1(0)$ , then

$$\begin{aligned} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{N+ps}} \frac{dxdy}{|x|^\beta |y|^\beta} &\geq \int_{\mathbb{R}^N \setminus B_4(0)} \int_{B_1(0)} \frac{1}{|x - y|^{N+ps}} \frac{dxdy}{|x|^\beta |y|^\beta} \\ &\geq \int_{\mathbb{R}^N \setminus B_4(0)} \frac{1}{(|y| + 4)^{N+ps}} \frac{dy}{|y|^\beta} \int_{B_1(0)} \frac{dx}{|x|^\beta} \\ &\geq C(N - \beta) \int_{\mathbb{R}^N \setminus B_4(0)} \frac{1}{(|y| + 4)^{N+ps}} \frac{dy}{|y|^\beta}. \end{aligned}$$

Since  $\beta < -ps$ , then  $\int_{\mathbb{R}^N \setminus B_4(0)} \frac{1}{(|y| + 4)^{N+ps}} \frac{dy}{|y|^\beta} = \infty$ . Hence we conclude.

### 3. APPLICATION

In this section we deal with the next problem

$$(3.35) \quad \begin{cases} L_{p,s} u &= \lambda \frac{u^{p-1}}{|x|^{ps}} + u^q, \quad u > 0 \text{ in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where

$$L_{s,p} u := \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+ps}} dy,$$

and  $0 < \lambda \leq \Lambda_{N,p,s}$ .

In the case where  $0 < q < p - 1$ , the existence result follows using variational arguments. More precisely we have:

- (1) If  $\lambda < \Lambda_{N,p,s}$ , then the existence of a solution  $u$  to (3.35) follows using classical minimizing argument. In this case  $u \in W_0^{s,p}(\Omega)$ .
- (2) If  $\lambda = \Lambda_{N,p,s}$ , the existence result follows using the improved Hardy inequality in Theorem 1.3. In this case  $u$  satisfies  $h_{s,\Omega}(u) < \infty$  where  $h_{s,\Omega}$  is defined by

$$(3.36) \quad h_{s,\Omega}(u) \equiv \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy - \Lambda_{N,p,s} \int_{\Omega} \frac{|u(x)|^p}{|x|^{ps}} dx.$$

This clearly implies that

$$\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+qs}} dxdy < \infty \text{ for all } q < p.$$

We deal now with the case  $q > p - 1$ .

Define  $w(x) = |x|^{-\gamma}$  with  $0 < \gamma < \frac{N-ps}{p-1}$ , then we have previously obtained that

$$L_{s,p}(w) = \Lambda(\gamma) \frac{w^{p-1}}{|x|^{ps}} \text{ a.e. in } \mathbb{R}^N \setminus \{0\},$$

where

$$\Lambda(\gamma) = \int_1^{+\infty} K(\sigma) (\sigma^\gamma - 1)^{p-1} \left( \sigma^{N-1-\gamma(p-1)} - \sigma^{ps-1} \right) d\sigma,$$

and  $K$  is given by (2.15). Let us begin by proving the next lemma.

**Lemma 3.1.** Assume that  $0 < \lambda < \Lambda_{N,p,s}$ , then there exist  $\gamma_1, \gamma_2$  such that

$$0 < \gamma_1 < \frac{N-ps}{p} < \gamma_2,$$

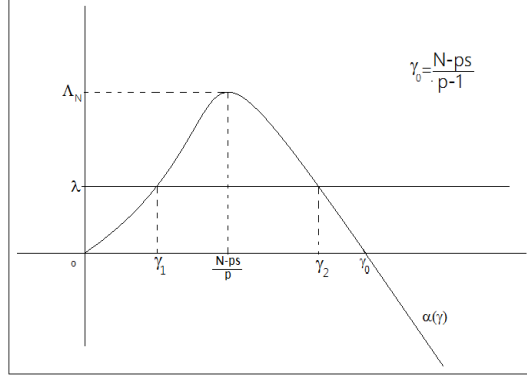
and  $\Lambda(\gamma_1) = \Lambda(\gamma_2) = \lambda$ .

*Proof.* We have  $\Lambda(0) = 0$ ,  $\Lambda(\frac{N-ps}{p}) = \Lambda_{N,p,s}$ ,  $\Lambda(\gamma) < 0$  if  $\gamma > \frac{N-ps}{p-1}$  and

$$\Lambda'(\gamma) = (p-1) \int_1^{+\infty} K(\sigma) \log(\sigma) (\sigma^\gamma - 1)^{p-2} \left( \sigma^{N-1-\gamma(p-1)} - \sigma^{ps+\gamma-1} \right) d\sigma.$$

It is clear that for  $\gamma_0 = \frac{N-ps}{p}$ , we have  $\Lambda'(\gamma_0) = 0$ ,  $\Lambda'(\gamma) > 0$  if  $\gamma < \gamma_0$  and  $\Lambda'(\gamma) < 0$  if  $\gamma > \gamma_0$ .

Hence, since  $\lambda < \Lambda_{N,p,s}$ , we get the existence of  $0 < \gamma_1 < \frac{N-ps}{p} < \gamma_2 < \frac{N-ps}{p-1}$  such that  $\Lambda(\gamma_1) = \Lambda(\gamma_2) = \lambda$ .  $\square$



Define  $q_+(p, s) = p - 1 + \frac{ps}{\gamma_1}$ , it is clear that  $p_s^* - 1 < q_+(p, s)$ . We have the next existence result.

**Theorem 3.2.** Assume that  $q < q_+(p, s)$ , then

- (1) If  $p - 1 < q < p_s^* - 1$ , problem (3.35) has a solution  $u$ . Moreover,  $u \in W_0^{s,p}(\Omega)$  if  $\lambda < \Lambda_{N,p,s}$  and  $h_{s,\Omega}(u) < \infty$  if  $\lambda = \Lambda_{N,p,s}$  where  $h_{s,\Omega}$  is defined in (3.36).
- (2) If  $p_s^* - 1 \leq q < q_+(p, s)$ , then problem (3.35) has a positive supersolution  $u$ .

*Proof.* Let us begin with the case where  $p - 1 < q < p_s^* - 1$ . If  $\lambda < \Lambda_{N,p,s}$ , then using the Mountain Pass Theorem, see [23], we get a positive solution  $u \in W_0^{s,p}(\Omega)$ . However, if  $\lambda = \Lambda_{N,p,s}$ , then using the improved Hardy inequality in Theorem 1.3 and the Mountain pass Theorem, we reach a positive solution  $u$  to problem (3.35) with  $h_{s,\Omega}(u) < \infty$ .

Assume now that  $p_s^* - 1 \leq q < q_+(p, s)$  and fix  $\lambda_1 \in (\lambda, \Lambda_{N,p,s})$  to be chosen later.

Let  $\gamma_1 \in (0, \frac{N-ps}{p})$  be such that  $\Gamma(\gamma_1) = \lambda_1$  and set  $w(x) = |x|^{-\gamma_1}$ , then

$$L_{s,p}(w) = \lambda_1 \frac{w^{p-1}(x)}{|x|^{ps}} \quad a.e \text{ in } \mathbb{R}^N \setminus \{0\}$$

with  $\frac{w^{p-1}}{|x|^{ps}} \in L_{loc}^1(\mathbb{R}^N)$ . Hence

$$L_{s,p}(w) = \lambda \frac{w^{p-1}(x)}{|x|^{ps}} + (\lambda_1 - \lambda) \frac{w^{p-1}(x)}{|x|^{ps}} \quad a.e \text{ in } \mathbb{R}^N \setminus \{0\}.$$

Using the fact that  $q < q_+(p, s)$ , we can choose  $\lambda_1 > \lambda$ , very close to  $\lambda$  such that  $\gamma_1(p-1) + ps > q\gamma_1$ , thus, in any bounded domain  $\Omega$ , we have

$$(\lambda_1 - \lambda) \frac{w^{p-1}(x)}{|x|^{ps}} \geq C(\Omega)w^q.$$

Define  $\hat{w} = Cw$ , by the previous estimates, we can choose  $C(\Omega) > 0$  such that  $\hat{w}$  will be a supersolution to (3.35) in  $\Omega$ . Hence the result follows.  $\square$

Now, we show the optimality of the exponent  $q_+(p, s)$ . We have the following non existence result.

**Theorem 3.3.** *Let  $q_+(p, s) = p-1+\frac{ps}{\gamma_1}$ , then if  $q > q_+(p, s)$ , then the unique nonnegative supersolution  $u \in W_{loc}^{s,p}(\Omega)$  to problem (3.35) is  $u \equiv 0$ .*

We first prove the next lemma which shows that the Hardy constant is independent of the domain.

**Lemma 3.4.** *Let  $\Omega \subset \mathbb{R}^N$  be a regular domain such that  $0 \in \Omega$ . Define*

$$\Lambda(\Omega) = \inf_{\{\phi \in C_0^\infty(\Omega) \setminus \{0\}\}} \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{N+ps}} dx dy}{\int_{\Omega} \frac{|\phi(x)|^p}{|x|^{ps}} dx},$$

then  $\Lambda(\Omega) = \Lambda_{N,p,s}$  defined in (1.3).

*Proof.* Recall that

$$\Lambda_{N,p,s} = \inf_{\{\phi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}\}} \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{N+ps}} dx dy}{\int_{\mathbb{R}^N} \frac{|\phi(x)|^p}{|x|^{ps}} dx},$$

thus  $\Lambda(\Omega) \geq \Lambda_{N,p,s}$ . It is clear that if  $\Omega_1 \subset \Omega_2$ , then  $\Lambda(\Omega_1) \geq \Lambda(\Omega_2)$ .

Now, using a dilatation argument we can prove that  $\Lambda(B_{R_1}(0)) = \Lambda(B_{R_2}(0))$  for all  $0 < R_1 < R_2$ . Hence we conclude that  $\Lambda(\Omega) \equiv \bar{\Lambda}$  does not depend of the domain  $\Omega$ .

For  $\phi \in C_0^\infty(\mathbb{R}^N)$ , we set

$$Q(\phi) \equiv \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{N+ps}} dx dy}{\int_{\mathbb{R}^N} \frac{|\phi(x)|^p}{|x|^{ps}} dx}.$$

Let  $\{\phi_n\}_n \subset C_0^\infty(\mathbb{R}^N)$  be such that  $Q(\phi_n) \rightarrow \Lambda_{N,p,s}$ . Without loss of generality and using a symmetrization argument we can assume that  $\text{Supp}(\phi_n) \subset B_{R_n}(0)$ . It is clear that  $Q(\phi_n) \geq \Lambda(\text{Supp}(\phi_n)) = \bar{\Lambda}$ , thus, as  $n \rightarrow \infty$ , it follows that  $\bar{\Lambda} \leq \Lambda_{N,p,s}$ . As a conclusion we reach that  $\bar{\Lambda} = \Lambda_{N,p,s}$  and the result follows.  $\square$

We need the next lemma.

**Lemma 3.5.** *Let  $\Omega$  be a bounded domain such that  $0 \in \Omega$ . Assume that  $u \in W^{s,p}(\mathbb{R}^N)$  is such that  $u \geq 0$  in  $\mathbb{R}^N$ ,  $u > 0$  in  $\Omega$  and  $L_{N,p,s}u \not\geq \lambda \frac{u^{p-1}}{|x|^{ps}}$  in  $\Omega$ , then there exists  $\eta > 0$  such that  $u(x) \geq C|x|^{-\gamma_1}$  in  $B_\eta(0)$  where  $\gamma_1$  is defined in Lemma 3.1.*

*Proof.* Without loss of generality we can assume that  $B_1(0) \subset \Omega$ .

Fixed  $\lambda < \Lambda_{N,p,s}$  and define

$$\tilde{w}(x) = \begin{cases} |x|^{-\gamma_1} - 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

It is clear that  $\tilde{w} \in W_0^{s,p}(B_1(0))$  and

$$(3.37) \quad \begin{cases} L_{p,s} \tilde{w} &= h(x) \frac{\tilde{w}^{p-1}}{|x|^{ps}} \text{ in } B_1(0), \\ \tilde{w} &= 0 \text{ in } \mathbb{R}^N \setminus B_1(0) \end{cases}$$

where

$$h(x) = \int_0^{\frac{1}{|x|}} |1 - \sigma^{-\tilde{\gamma}}|^{p-2} (1 - \sigma^{-\tilde{\gamma}}) \sigma^{N-1} K(\sigma) d\sigma + (1 - |x|^{\tilde{\gamma}}) \int_{\frac{1}{|x|}}^{\infty} \sigma^{N-1} K(\sigma) d\sigma.$$

Using the definition of  $\gamma_1$ , see Lemma 3.1, we can prove that  $h(x) \leq \lambda$  for all  $x \in B_1(0)$ .

Since  $L_{p,s}u \geq 0$  and  $u > 0$  in  $\Omega$ , then using the nonlocal weak Harnack inequality in [11], we get the existence of  $\varepsilon > 0$  such that  $u \geq \varepsilon$  in  $\bar{B}_1(0)$ .

Therefore we obtain that

$$(3.38) \quad \begin{cases} L_{p,s} u &\geq \lambda \frac{u^{p-1}}{|x|^{ps}} \text{ in } B_1(0) \\ L_{p,s} \tilde{w} &\leq \lambda \frac{\tilde{w}^{p-1}}{|x|^{ps}}, \text{ in } B_1(0), \\ u &\geq \tilde{w} \text{ in } \mathbb{R}^N \setminus B_1(0). \end{cases}$$

Thus by the comparison principle in Lemma 2.4, it follows that  $\tilde{w} \leq u$  which is the desired result.  $\square$

We are now in position to prove Theorem 3.3.

**Proof of Theorem 3.3.** We argue by contradiction. Assume the existence of  $u \not\geq 0$  such that  $u \in W^{s,p}(\mathbb{R}^N)$  and  $u$  is a supersolution to problem (3.35) in  $\Omega$ , then  $u > 0$  in  $\Omega$ . Let  $\phi \in \mathcal{C}_0^\infty(B_\eta(0))$  with  $B_\eta(0) \subset \subset \Omega$  and  $\eta > 0$  to be chosen later.

Using Picone's inequality in Lemma 2.3, it follows that

$$\|\phi\|_{X_0^{s,p}(\mathbb{R}^N)}^p \geq \int_{B_\eta(0)} \frac{L_{p,s}(u)}{u^{p-1}} |\phi|^p dx.$$

Thus

$$\|\phi\|_{X_0^{s,p}(\mathbb{R}^N)}^p \geq \int_{B_\eta(0)} u^{q-(p-1)} |\phi|^p dx.$$

Since  $q > q_+(p, s)$ , we get the existence of  $\varepsilon > 0$  such that

$$(\gamma_1 - \varepsilon)(q - (p - 1)) > ps + \rho$$

for some  $\rho > 0$ . Thus, using Lemma 3.5, we can choose  $\eta > 0$  such that

$$u^{q-(p-1)} \geq C|x|^{-ps-\rho} \text{ in } B_\eta(0).$$

Therefore

$$\|\phi\|_{X_0^{s,p}(\mathbb{R}^N)}^p \geq C \int_{B_\eta(0)} \frac{|\phi|^p}{|x|^{ps+\rho}} dx,$$

which is a contradiction with the optimality of the Hardy inequality proved in Lemma 3.4. Hence we conclude.  $\blacksquare$

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